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ON SUBSEQUENCES OF THE HAAR SYSTEM IN $C(\Delta)$

BY

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ABSTRACT

Spaces arising as spans of subsequences of the Haar system in $C(\Delta)$ are studied. It is shown that for any compact metric space H there is a subsequence whose span is isomorphic to $C(H)$, yet that subsequences exist whose spans are not $\mathcal{L}_\infty$ spaces.

1. In [1], Gamlen and Gaudet showed that only two Banach spaces, l_p and L_p, arise as the spans of subsequences of the Haar System in L_p, $1 < p < \infty$. In this paper we investigate spans of subsequences of the Haar system in $C(\Delta)$, where Δ denotes the Cantor set. The situation is somewhat different. In section 2 we show that for any compact metric space H, there is a subsequence of the Haar system whose span is isomorphic to $C(H)$, while in section 4 we give an example of a subsequence whose span is not an $\mathcal{L}_\infty$ space. In section 3 we show that for all $A \subset N$, $[\{\varphi_n\}_{n\in A}]$ contains c_0, and provide a sufficient condition for $[\{\varphi_n\}_{n\in A}]$ to contain an isomorph of $C(\Delta)$.

Our notation is standard. If C is a subset of a Banach space X, we use $[C]$ to denote the closed linear span of C. We write $X \sim Y$ (resp. $X \approx Y$) to denote that X is isomorphic (resp. isometric) to Y.

The Haar system is a monotone basis for $C(\Delta)$ and may be defined as follows. Let $\mathcal{D} = \{\Delta_{n,i} : n = 0, 1, \cdots; 0 \leqq i < 2^n\}$ be a basis of clopen sets for the topology of Δ such that

(i) $\Delta_{0,0} = \Delta$,

(ii) $\Delta_{n,i} \cap \Delta_{n,j} = \emptyset$ if $i \neq j$,

(iii) $\Delta_{n+1,2i} \cup \Delta_{n+1,2i+1} = \Delta_{n,i}$,

and define $\varphi_0 = \chi_{\Delta_{0,0}}$, $\varphi_{2^n+i} = \chi_{\Delta_{n+1,2i}} - \chi_{\Delta_{n+1,2i+1}}$ for $n = 0, 1, \cdots$; $0 \leqq i < 2^n$.

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The author would like to thank the referee for his helpful comments. In particular, Theorem 2.1 below is somewhat stronger than the original theorem. The proof of Theorem 2.1 presented here, which is simpler than the original proof, is due to the referee.

2. In this section we show that for any compact metric space K, there is a subsequence of the Haar system whose span is isomorphic to $C(K)$. If K is uncountable, this follows from Milutin's Theorem [4, p. 174], and if K is countable, then K is homeomorphic to a closed subset of the Cantor set.

THEOREM 2.1. *For any closed subset $K \subset \Delta$, there exists a subsequence $\{\varphi_{n_k}\}$ such that $X = [\{\varphi_{n_k}\}] \approx C(K)$ and X is complemented in $C(\Delta)$ by a projection of norm one.*

PROOF. Let R denote the restriction operator from $C(\Delta)$ onto $C(K)$, let $A = \{n : n = 0 \text{ or } \varphi_n \text{ is not constant on } K\}$, and let $X = [\{\varphi_n\}_{n \in A}]$. Then, if $n \neq m$, $(R\varphi_n)\,(R\varphi_m)$ is either 0 or $R\varphi_{\max(n,m)}$, and an induction argument shows that $(R\varphi_{2^j+i})^2 = R\chi_{\Delta j,i}$. Thus $R(X)$ is a separating subalgebra of $C(K)$. By the Stone–Weierstrass theorem, $R\,|_X$ is an isometry onto $C(K)$. Furthermore, X is complemented by the projection $P = (R\,|_X)^{-1}R$.

3. In this section we show that any space X arising as the span of a subsequence of the Haar system contains an isomorph of c_0, and give a sufficient condition for X to contain an isometric isomorph of $C(\Delta)$.

THEOREM 3.1. *If $X = [\{\varphi_{n_k}\}]$, then X contains a subspace isomorphic to c_0.*

PROOF. We show that $\{\varphi_{n_k}\}$ contains a subsequence equivalent to either (i) the unit vector basis in c_0 or (ii) the basis $\{x_n\}$ for c defined by $x_n(i) = 0,\ i < n$ and $x_n(i) = 1,\ i \geqq n$.

If $\{\varphi_{n_k}\}$ contains a subsequence $\{\psi_m\}$ of disjointly supported functions, then (i) holds. Otherwise, there is a subsequence $\{\psi_m\}$ such that $\operatorname{supp} \psi_{m+1} \subset \operatorname{supp} \psi_m$. We may assume, in fact, that $\operatorname{supp} \psi_{m+1} \subset \psi_m^{-1}(1)$. Then for any scalar sequence $\{a_i\}_{i=1}^N$,

$$\left\| \sum a_i x_i \right\| = \max_j \left\{ \left| \sum_{i=1}^{j} a_i \right| \right\}$$

$$\leqq \max_j \left(\max \left\{ \left| \sum_{i=1}^{j} a_i \right|, \left| \sum_{i=1}^{j-1} a_i - a_j \right| \right\} \right)$$

$$= \left\| \sum a_i \psi_i \right\|$$

$$\leqq 3 \max_j \left\{ \left| \sum_{i=1}^{j} a_i \right| \right\}$$

$$= 3 \left\| \sum a_i x_i \right\|,$$

so that (ii) holds.

It is easy to see that in fact each $X = [\{\varphi_{n_k}\}]$ contains a subspace isometric to c_0.

The following is a special case of a lemma of Lindenstrauss and Pelczynski [3]. We use $[\cdot]$ to denote the greatest integer function.

LEMMA 3.2. *Let $\{f_n\}_{n=0}^{\infty} \subset C(\Delta)$ be a sequence of $\{-1,0,1\}$-valued functions. Let $A_{0,0} = f_0^{-1}(1)$, and $A_{n,i} = f_{2^{n-1}+[i/2]}^{-1}((-1)^i)$ for $n > 0$ and $i = 0, \cdots, 2^n - 1$. Assume that*

$$A_{n,i} \cap A_{n,j} = \varnothing \quad \text{whenever } i \neq j, \quad \text{and} \tag{5}$$

$$A_{n+1,2i} \cup A_{n+1,2i+1} \subset A_{n,i}. \tag{6}$$

Then $\{f_n\}$ is isometrically equivalent to the Haar system.

THEOREM 3.3. *Let $X = [\{\varphi_{n_k}\}]$ and suppose $T = \{t\colon t \in \operatorname{supp} \varphi_{n_k}$ for infinitely many $k\}$ contains a subset T_1 dense in itself. Then X contains a subspace isometric to $C(\Delta)$.*

PROOF. We inductively construct a sequence $\{\psi_n\} \subset X$ satisfying the hypotheses of the preceding lemma. Select $t_0 \in T_1$ and k_0 such that $\varphi_{n_{k_0}}(t_0) \neq 0$. We may assume without loss of generality that $\varphi_{n_{k_0}}(t_0) = 1$, and let $\psi_0 = \varphi_{n_{k_0}}$. Since T_1 is dense in itself, there exist distinct $t_1, t_2 \in T_1 \cap \psi_0^{-1}(1)$ and $k_1, k_2 > k_0$ such that

$$\operatorname{supp} \varphi_{n_{k_1}} \cap \operatorname{supp} \varphi_{n_{k_2}} = \varnothing, \tag{7}$$

$$\operatorname{supp} \varphi_{n_{k_1}} \cup \operatorname{supp} \varphi_{n_{k_2}} \subset \psi_0^{-1}(1), \tag{8}$$

$$t_1 \in \operatorname{supp} \varphi_{n_{k_1}}, \qquad t_2 \in \operatorname{supp} \varphi_{n_{k_2}}. \tag{9}$$

Define

$$\psi_1(t) = \varphi_{n_{k_1}}(t_1)\varphi_{n_{k_1}}(t) - \varphi_{n_{k_2}}(t_2)\varphi_{n_{k_2}}(t).$$

From (8) and (9) it follows that $\psi_1(t_1) = 1$, $\psi_1(t_2) = -1$, and $\operatorname{supp} \psi_1 \subset \psi_0^{-1}(1)$. Suppose $\{-1,0,1\}$-valued functions $\psi_0, \cdots, \psi_{2^n-1}$ have been constructed so that

$$T_1 \cap \psi_k^{-1}(1) \neq \varnothing, \qquad T_1 \cap \psi_k^{-1}(-1) \neq \varnothing, \tag{10}$$

$$\operatorname{supp} \psi_{2^h+j} \subset \psi^{-1}_{2^{h-1}+[j/2]}((-1)^j), \qquad \text{and} \tag{11}$$

$$\operatorname{supp} \psi_{2^h+j} \cap \operatorname{supp} \psi_{2^h+i} = \varnothing \quad \text{whenever } i \neq j. \tag{12}$$

For $0 \leqq j < 2^n$, construct ψ_{2^n+j} as follows. By (10) and the assumption on T_1, select distinct $t_1, t_2 \in \psi^{-1}_{2^{n-1}+[j/2]}((-1^j)) \cap T_1$, and choose k_1, k_2 such that

$$t_i \in \operatorname{supp} \varphi_{n_{k_i}} \quad \text{for } i = 1, 2;$$

$$\operatorname{supp} \varphi_{n_{k_1}} \cap \operatorname{supp} \varphi_{n_{k_2}} = \varnothing, \qquad \text{and}$$

$$\operatorname{supp} \varphi_{n_{k_1}} \cup \operatorname{supp} \varphi_{n_{k_2}} \subset \psi^{-1}_{2^{n-1}+[j/2]}((-1)^j).$$

Define

$$\psi_{2^n+j}(t) = \varphi_{n_{k_1}}(t_1)\varphi_{n_{k_1}}(t)\varphi_{n_{k_2}}(t_2)\varphi_{n_{k_2}}(t).$$

Then $\{\psi_k\}$ is a sequence of $\{-1, 0, 1\}$-valued functions satisfying the hypotheses of Lemma 3.2. Hence $[\{\psi_n\}] \approx C(\Delta)$.

4. In this section we construct a subsequence whose span is not an $\mathcal{L}_\infty$ space. Parts of the exposition are simplified by working not with $C(\Delta)$ directly, but with the Banach space D, defined to be the closed span in $L^\infty(0,1)$ of the characteristic functions of diadic intervals, and with Haar functions defined using sets $\Delta_{n,i} = (i/2^n, (i+1)/2^n)$. We denote the sequence biorthogonal to the Haar system by $\{\varphi^*_n\}$, and the levels of the Haar system by $H_0 = \{\varphi_0\}$ and $H_j = \{\varphi_{2^{j-1}+i}\colon 0 \leqq i < 2^{j-1}\}$ for $j > 0$.

LEMMA 4.1. *Let $X_n = [\bigcup_{j=0}^n H_{2j}]$ and $Y_n = [\bigcup_{j=0}^{2n} H_j] \approx l_\infty^{2^{2n}}$. Let $P\colon Y_n \to X_n$ be a projection. Then $\|P\| \geqq (2n+1)/3$.*

PROOF. For each $k < 2^{2n}$, let g_k be the isometry of Y_n induced by interchanging the sets $\varphi_k^{-1}(1)$ and $\varphi_k^{-1}(-1)$. Specifically, if $k = 2^j + i$,

$$(g_{2^j+i}f)(t) = \begin{cases} f(t) & t \notin \Delta_{j,i}, \\ f(t+2^{-j-1}) & t \in \Delta_{j+1,2i}, \\ f(t-2^{-j-1}) & t \in \Delta_{j+1,2i+1}. \end{cases}$$

Let G be the group of isometries generated by $\{g_k\colon k < 2^{2n}\}$. Then $GX_n = X_n$, so

$$Q = \frac{1}{\operatorname{card} G} \sum_{g \in G} g^{-1}Pg \tag{13}$$

is a projection of Y_n onto X_n which satisfies

$$\|Q\| \leqq \|P\|, \qquad \text{and} \tag{14}$$

(15) $$gQ = Qg, \qquad \text{for all } g \in G.$$

Notice that

(16) $$j > k \text{ or } \varphi_j \cdot \varphi_k = 0 \text{ implies } g_j\varphi_k = \varphi_k,$$

and

(17) $$\varphi_k^*(g_k f) = -\varphi_k^*(f) \qquad \text{for all } f \in Y_n.$$

We investigate $Q\varphi_k$ for $\varphi_k \in H_{2j-1}$. Let $\varphi_i \in X_n$. If $i > k$ or $\varphi_i \cdot \varphi_k = 0$, then (16) implies

$$Qg_i\varphi_k = Q\varphi_k,$$

so

$$Q\varphi_k = g_iQ\varphi_k$$

by (15). Thus by (17)

(18) $$\varphi_i^*(Q\varphi_k) = -\varphi_i^*(Q\varphi_k) = 0.$$

If $i < k$ and $\operatorname{supp}\varphi_k \subset \operatorname{supp}\varphi_i$, then $g_kQ\varphi_k = Qg_k\varphi_k = -Q\varphi_k$, so

(19) $$\varphi_i^*(Q\varphi_k) = \varphi_i^*(g_kQ\varphi_k) = -\varphi_i^*(Q\varphi_k) = 0.$$

By (18) and (19), $Q\varphi_k = 0$ whenever $\varphi_k \notin X_n$. Let $f = \varphi_0 + 2\sum_{i=1}^{2n}(-1)^i\varphi_{2^i}$. Then $\|f\| = 3$, but $\|Qf\| = \|\varphi_0 + 2\varphi_2 + 2\varphi_8 + \cdots + 2\varphi_{2^{2n-1}}\| = 2n+1$. Hence $\|Q\| \geqq (2n+1)/3$, and the lemma follows from (14).

THEOREM 4.2. *There exists a subsequence of the Haar system whose span is not an $\mathcal{L}_\infty$ space.*

PROOF. Define subspaces X_n' (resp. Y_n') of D by $f \in X_n'$ (resp. Y_n') iff $\operatorname{supp} f \subset \Delta_{n,1}$ and $f(2^n(t-2^{-n})) \in X_n$ (resp. Y_n). Define $X = [\bigcup_{n=1}^\infty X_n']$, $Y = [\bigcup_{n=1}^\infty Y_n']$. Then X is obviously the span of a subsequence of the Haar system.

Let P_n be the norm one projection of X onto X_n' defined by $P_nf = \chi_{\Delta_{n,1}} \cdot f$. If X were complemented in Y, say by a projection P, then for each n, $Q_n = P_n \circ (P \mid Y_n')$ would be a projection from Y_n' onto X_n' with $\|Q_n\| \leqq \|P\|$. But for sufficiently large n, this contradicts Lemma 4.1. Thus X is non-complemented in Y, and since Y is separable, X is not isomorphic to c_0. Now $Y \sim c_0$, and by [2], all $\mathcal{L}_\infty$ subspace of c_0 are in fact isomorphic to c_0. Hence X is not an $\mathcal{L}_\infty$ space.

Since all $C(K)$ spaces are $\mathcal{L}_\infty$ spaces, the space X above is an example of the span of a subsequence of the Haar system which is not isomorphic to a $C(K)$ space.

References

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